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EVALUATION OF BULK VELOCITY AND TEMPERATURE FOR TURBULENT FLOW IN TUBES

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Abstract—This article describes two methods by which bulk velocities and temperatures can be found without resorting to traversing or the use of mixing chambers; in each case by only one local measurement at the section considered. Both methods depend on a knowledge of the velocity and temperature profiles at that section; knowledge which is now available for fully established turbulent flow in smooth tubes.

(a) *Axial method:* The velocity or temperature probe is placed at the axis, and the reading multiplied by an appropriate weighting factor.

(b) *y_m/r method:* The velocity or temperature probe is placed a distance from the wall at which it reads the mean value directly.

Information is presented for finding the bulk velocity and temperature by either method, and their relative merits are discussed.

When making temperature measurements in a fluid with a thermocouple it is often convenient to bring the wire out radially from a butt-welded junction, and it is then necessary to establish that no serious conduction error is introduced. Data are presented which allow estimates to be made of temperature errors for a centrally placed junction of various materials and gauges, and for various rates of flow and types of fluid including liquid metals.

Résumé—Cet article décrit deux méthodes permettant le calcul des vitesses et des températures moyennes sans nécessiter de dispositif permettant le déplacement transversal et sans utiliser de chambre de mélange; dans chaque cas, la mesure se fait en un point seulement de la section considérée. Les deux méthodes dépendent de la connaissance des profils de vitesses et de températures dans cette section; cette connaissance est maintenant valable pour un écoulement turbulent pleinement établi dans des tubes lisses.

(a) *Méthode axiale*: la sonde de vitesse ou de température est placée sur l'axe et la lecture est multipliée par un facteur de pondération convenable.

(b) *Méthode y_m/r*: la sonde de vitesse ou de température se trouve à une distance de la paroi telle que la lecture fournit directement la valeur moyenne.

Des indications sont données pour le calcul des vitesses et températures moyennes par les deux méthodes dont les qualités respectives sont discutées.

Quand on mesure les températures dans un fluide avec un thermocouple, il est souvent commode de sortir le fil radialement grâce à une connexion soudée et il est alors nécessaire d'établir que l'erreur de conduction introduite est négligeable. On présente ici des résultats permettant d'estimer l'erreur de température pour une disposition centrale des jonctions de fils et de jauges de nature variée, pour différents régimes d'écoulement et pour différentes sortes de fluides, y compris les métaux liquides.

Zusammenfassung—Dieser Aufsatz beschreibt zwei Verfahren zur Ermittlung von mittleren Geschwindigkeiten und Temperaturen ohne Messung der Geschwindigkeits- und Temperaturverteilung und ohne die Anwendung von Mischern. In jedem Falle hängt die Auswertung für einen Rohrquerschnitt von einer örtlichen Messung ab. Für beide Methoden braucht man die Kenntnis der Temperatur und Geschwindigkeitsprofile in diesem Querschnitt, wie sie für ausgebildete turbulente Strömung in glatten Rohren bekannt ist.

(a) *Axiales Verfahren*: Die Geschwindigkeits- oder Temperatursonde wird in die Rohrachse gelegt, und der Messwert wird mit einem passenden Faktor multipliziert.

(b) *y_m/r Verfahren*: Die Geschwindigkeits- oder Temperatursonde wird in einem Wandabstand gelegt, wo sie den Mittelwert direkt angibt.

Für beide Verfahren werden die notwendigen Angaben gemacht, um die mittlere Geschwindigkeit und Temperatur zu erhalten und ihre Vorzüge werden miteinander verglichen.

Bei Messung von Flüssigkeitstemperaturen mit einem Thermoelement werden oft die Drähte radial von der Lötstelle nach aussen geführt und es ist dann wichtig festzustellen, dass keine grossen Messfehler durch Wärmeleitung entstehen. Zur Abschätzung von Temperaturmessfehlern für axiale Lötstellen aus verschiedenen Drahtmaterialien und Durchmessern und für verschiedene Strömungsgeschwindigkeiten und Flüssigkeiten, einschliesslich flüssiger Metalle, werden Zahlenwerte angegeben.

Abstract—В статье описываются два метода, с помощью которых можно определить осредненные по объёму скорости и температуры, не прибегая к перемещению соответствующих датчиков или использованию камер смешения. В каждом случае они сводятся только к одному локальному замеру в рассматриваемом сечении. Оба метода требуют знания профилей скорости и температуры в этом сечении. Для полностью установившегося турбулентного потока в гладких трубах эти профили хорошо известны.

(*а*) *Осевой метод:* Датчик скорости или температуры помещается на оси, а его показание умножается на соответствующий коэффициент веса.

(*б*) *Метод* y_m/r : Датчик скорости или температуры помещается на расстоянии от стенки, на котором он непосредственно даёт среднее значение.

Даётся описание обоих методов и рассматриваются их относительные преимущества.

При измерениях температуры в потоке с помощью термопары часто бывает удобно вывести проводники от спая по радиусам. В этом случае необходимо убедиться, что не вносится серьёзная погрешность, связанная с теплопроводностью проводников. Приведены данные, позволяющие оценить температурные погрешности для различных по материалу и диаметру проводов термопар с центральным расположением спая, при различных скоростях потока и различных субстанциях, включая жидкие металлы.

In experimental heat transfer work it is often necessary to measure bulk velocities and temperatures of fluids flowing through tubes. Bulk velocity can be obtained either by radial traversing (e.g. with pitot or hot wire probe) or by some kind of orifice meter. Bulk temperature can be obtained by radial traversing or by means of a mixing chamber. (It must be remembered that when a bulk temperature is obtained by traversing, both velocity and temperature traverses are necessary if the correct weighted-mean value is to be obtained.) Orifice meters or mixing chambers, while giving bulk values directly, can only be used at entry to and exit from a test length, since they would completely upset the flow when mounted at an intermediate section. Traverses are impractical in tubes of small diameter, and are at best laborious, and they cannot normally be justified unless the purpose of an investigation is the study of the velocity and temperature profiles themselves.

This article describes two further methods by which bulk velocities and temperatures can be found; in each case by only one local measurement at the section considered. Both methods depend on a knowledge of the velocity and temperature profiles at that section: knowledge which is now available for fully established turbulent flow in smooth tubes.

(a) *Axial method.* The velocity or temperature probe is placed at the axis, and the reading multiplied by an appropriate weighting factor u_m/u_r or $(t_m - t_w)/(t_r - t_w)$, respectively.

(b) y_m/r *method.* The velocity or temperature probe is placed a distance from the wall y_{mu} or y_{mt}, respectively, at which it reads the mean value directly.

Information is presented in Sections 1 and 2 for finding bulk velocity and temperature by either method, and their relative merits are discussed.

When making temperature measurements in a fluid with a thermocouple, the wire is often laid axially along an isothermal, before being taken out radially, to reduce conduction error. Sometimes it is more convenient, however, to bring the wire out radially from a butt-welded junction, and it is then necessary to establish that no serious conduction error is introduced. Data are presented in Section 3 which allow estimates to be made of temperature errors for a centrally placed junction of various wire materials and gauges, and for various rates of flow and types of fluid including liquid metals.

NOMENCLATURE

d = thermocouple wire diameter;

D = tube diameter;

h = convection heat transfer coefficient;
I_1 = integral defined by equations (3) and (4);
I_2 = integral defined by equations (11) and (12);
q = heat flow per unit area and unit time;
r = tube radius;
r^+ = dimensionless radius $(= r\sqrt{(\tau_w/\rho)}/\nu)$;
t = temperature at any radius;
Δt = temperature relative to wall $(= t - t_w)$;
Δt^+ = dimensionless temperature relative to wall $(= \Delta t \rho c_p \sqrt{(\tau_w/\rho)}/q_w)$;
u = velocity at any radius;
u^+ = dimensionless velocity $\{= u/\sqrt{(\tau_w/\rho)}\}$;
y = distance from wall;
y^+ = dimensionless distance from wall $(= y\sqrt{(\tau_w/\rho)}/\nu)$;
y_{mt}, y^+_{mt} = wall distance at which $t = t_m$;
y_{mu}, y^+_{mu} = wall distance at which $u = u_m$.

Greek symbols
μ = dynamic viscosity;
ν = kinematic viscosity;
ρ = density;
τ = shear stress.

Dimensionless groups
P = Prandtl number $(= c_p\mu/k)$;
R_d = Reynolds number based on thermocouple wire diameter $(= ud/\nu)$;
R_m = Reynolds number based on mean velocity $(= 2u_m r/\nu)$;
R_r = Reynolds number based on axial velocity $(= 2u_r r/\nu)$;
ϵ_M/ϵ_H = ratio of eddy diffusivities for momentum and heat transfers.

Suffixes
d = refers to the thermocouple wire;
f = refers to the fluid;
m = mean value;
r = at axis of tube;
w = at wall.

1. MEASUREMENT OF MEAN VELOCITY

1.1. *Axial method*

For a tube, the mean or bulk velocity is defined by the equation

$$\rho\, u_m\, \pi r^2 = \rho \int_0^r u\, 2\pi(r - y)dy \qquad (1)$$

in which it is assumed that ρ is independent of y. Hence the ratio of mean to axial velocity is obtained as

$$\frac{u_m}{u_r} = \frac{2}{u_r\, r^2}\int_0^r u(r - y)dy \qquad (2)$$

To find the integral, it is necessary to know the function $u = f(y)$. It has been found [1] that if the velocity and distance from the wall are expressed in dimensionless form by $u^+ = u/\sqrt{(\tau_w/\rho)}$ and $y^+ = y\sqrt{(t_w/\rho)}/\nu$, respectively, it is possible to deduce a universal velocity profile, Fig. 1, which fits experimental results over a wide range of Reynolds numbers. Introducing u^+ and y^+ into equation (2), the ratio u_m/u_r becomes

$$\frac{u_m}{u_r} = \frac{u_m^+}{u_r^+} = \frac{2}{u_r^+ r^{+2}}\int_0^{r+} u^+ (r^+ - y^+)dy^+ = \frac{2}{u_r^+ r^{+2}} I_1 \qquad (3)$$

Using the equatons in Fig. 1, I_1 can be expanded to give

$$I_1 = \int_0^5 y^+(r^+ - y^+)dy^+ + \int_5^{30}(-3{\cdot}05 + 5 \ln y^+)(r^+ - y^+)dy^+ + \int_{30}^{r+}(5{\cdot}5 + 2{\cdot}5 \ln y^+)(r^+ - y^+)\, dy^+ = 1{\cdot}25\, r^{+2}(0{\cdot}7 + \ln r^+) - 63{\cdot}9\, r^+ + 573 \qquad (4)$$

In this way values of u_m/u_r can be found for various values of r^+.

It is normally more convenient to relate u_m/u_r to the Reynolds number based either on axial velocity $R_r = 2u_r r/\nu$, or on mean velocity $R_m = 2u_m r/\nu$. It can easily be shown that

$$R_r = 2u_r^+ r^+ \qquad (5)$$

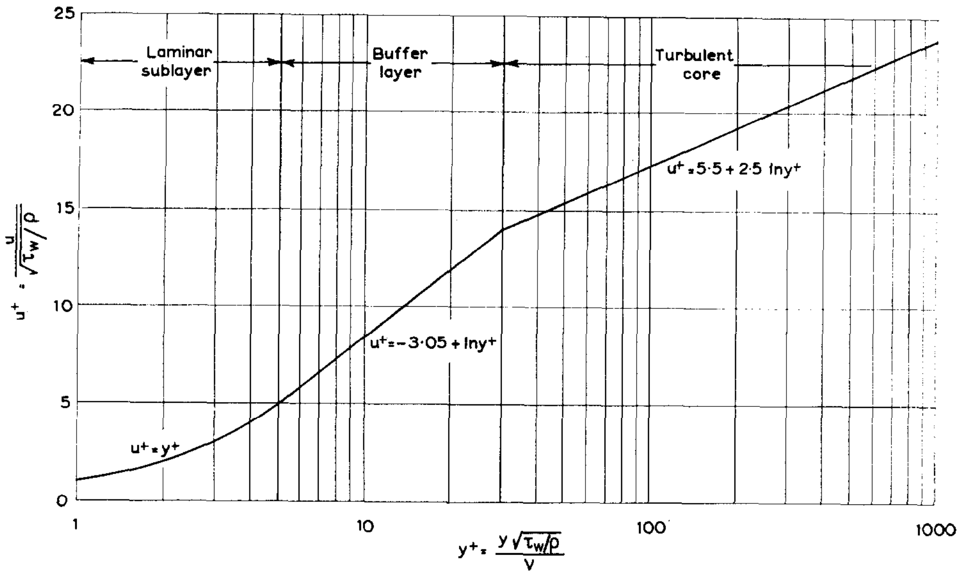

FIG. 1. Universal velocity profile.

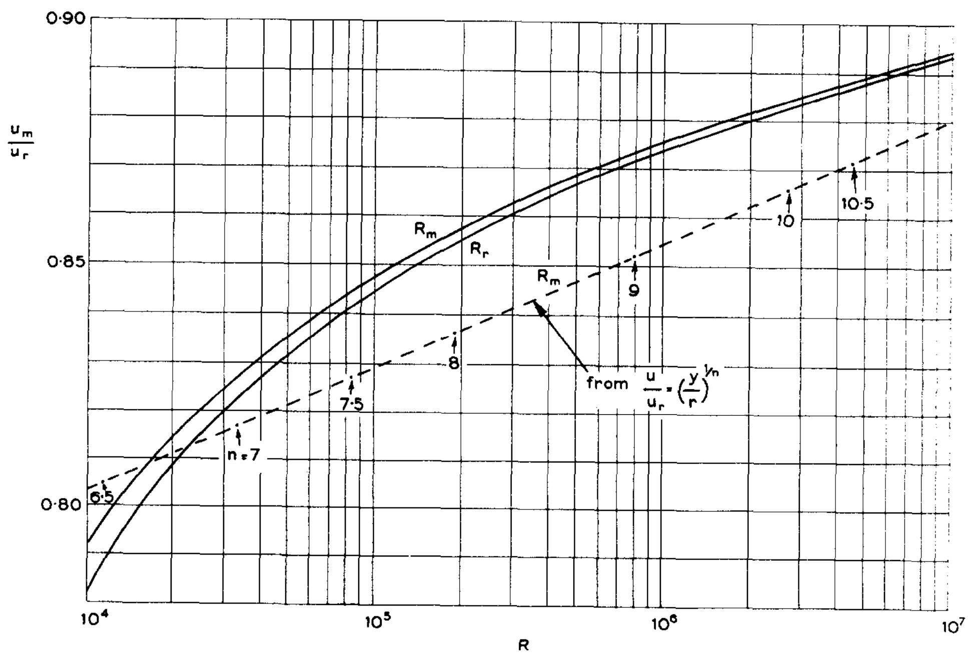

FIG. 2. Ratio u_m/u_r against Reynolds number $R_r = 2u_r r/\nu$ and $R_m = 2u_m r/\nu$.

and

$$R_m = 2u_m^+ r^+ = R_r \frac{u_m}{u_r} \tag{6}$$

Thus with these equations, and u^+ taken from Fig. 1, it is possible to relate u_m/u_r with R_r or R_m: these relations are shown in Fig. 2. If the axial velocity u_r is measured in an experiment, it is then possible to calculate the value of R_r, and the appropriate mean velocity $u_m = u_r\ (u_m/u_r)$ can be found by means of Fig. 2.

It is interesting to compare the result with that obtained for fully developed laminar flow when a parabolic velocity profile can be assumed: the value of u_m/u_r is then 0·5 and is independent of Reynolds number.

1.2. y_{mu}/r *method*

The mean velocity can also be found by placing the probe at the appropriate wall distance y_{mu}. For various values of r^+, corresponding values of u_m^+ have been obtained from equation (3) and corresponding values of R_m from (6). It is found that y_{mu} is always in the turbulent core, and to find it, it is necessary to use the appropriate equation from Fig. 1. Thus

$$y_{mu}^+ = \exp\left(\frac{u_m^+ - 5{\cdot}5}{2{\cdot}5}\right) \tag{7}$$

and hence the correct placing of the velocity probe is found as

$$\frac{y_{mu}}{r} = \frac{y_{mu}^+}{r^+} = \frac{1}{r^+}\exp\left(\frac{u_m^+ - 5{\cdot}5}{2{\cdot}5}\right)$$

Values of y_{mu}/r against R_m are plotted in Fig. 3.

Again for comparison, the value of y_{mu}/r for fully developed laminar flow is 0·293.

1.3. *Conclusions*

Mean velocity deduced from an axial measurement is to be preferred to that deduced from a measurement at y_{mu}. Firstly, at the axis a larger and more easily read deflexion of the measuring instrument is obtained and wall interference is less serious. Secondly, near the axis u does not vary appreciably with y, and neither location nor instrument width are as critical. The advantage of the y_{mu} method is that several pitot tubes, connected by a piezometer ring, can be spaced around the circumference so that the effect of any slight asymmetry in the flow is minimized.

How critical the location is with the y_{mu}/r method can be shown by the following argument. From the velocity profile, $u^+ = 5{\cdot}5 + \ln y^+$, in the turbulent core,

$$\frac{du^+}{dy^+} = \frac{2{\cdot}5}{y^+}$$

or

$$\frac{dy^+}{r^+} = 0{\cdot}4\,\frac{du^+}{u^+}\frac{y^+}{r^+}\,u^+$$

Putting $y^+ = y_{mu}^+$, $u^+ = u_m^+$, and placing a limit of 1 per cent on $\delta u^+/u_m^+$, the maximum permissible deviation is found to be

$$\frac{\delta y}{r} = \frac{\delta y^+}{r^+} = 0{\cdot}004\,\frac{y_{mu}^+}{r^+}\,u_m^+ \tag{8}$$

The results are plotted in Fig. 3.

Jakob [2] presented an analysis similar to that in Sections 1.1 and 1.2, but based on the simpler and earlier velocity profile $u/u_r = (y/r)^{1/n}$ where n is a function of Reynolds number. The results, for comparison with the curves based on the universal velocity profile, have been added as dotted curves in Figs. 2 and 3.

2. MEASUREMENT OF MEAN TEMPERATURE

2.1. *Axial method*

For a tube, the mean or bulk temperature is defined by the equation

$$\rho c_p\, u_m\,(t_m - t_w)\,\pi r^2 = \rho c_p \int_0^r u(t - t_w)\,2\pi\,(r - y)dy \tag{9}$$

in which it is assumed that ρ and c_p are independent of y, and u_m is given by equation (3). Writing Δt for $(t - t_w)$, the ratio of mean to axial temperature becomes

$$\frac{\Delta t_m}{\Delta t_r} = \frac{2}{u_m \Delta t_r r^2}\int_0^r u\Delta t\,(r - y)dy \tag{10}$$

To evaluate the integral, it is necessary to know the functions $u = f(y)$ and $t = g(y)$. For the velocity profile it is again possible to use the dimensionless presentation of Fig. 1, while for

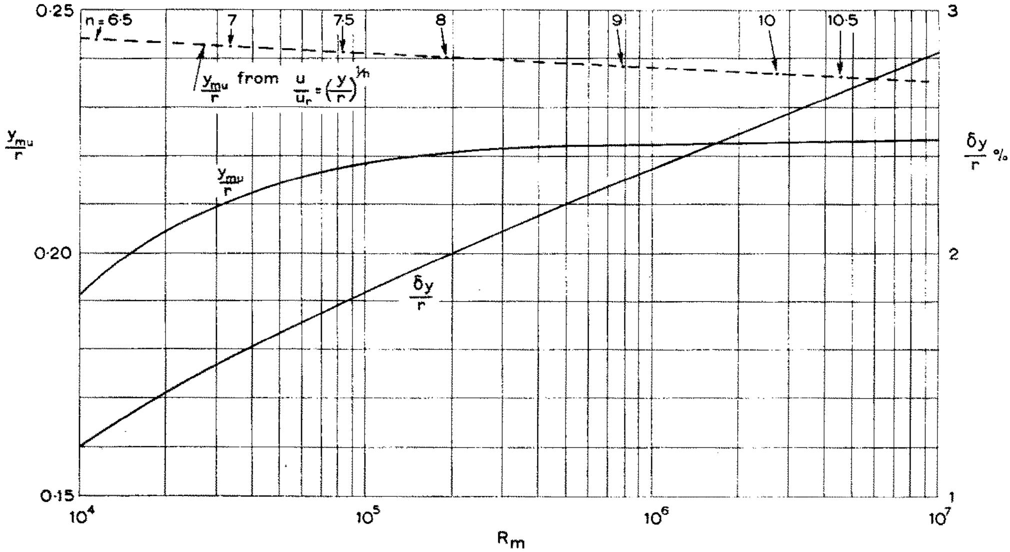

FIG. 3. Position y_{mu}/r of velocity probe and permissible location error $\delta y/r$ (for 1 per cent error in u_m) against R_m.

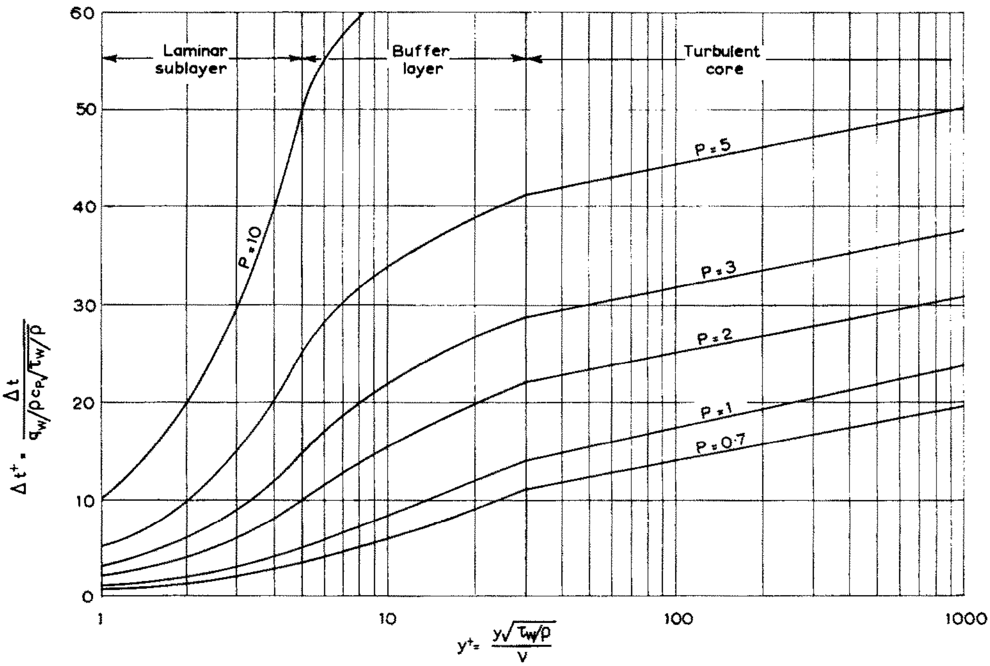

FIG. 4. Universal temperature profile.

Laminar sublayer $\Delta t^+ = Py^+$;

Buffer layer $\Delta t^+ = 5[P + \ln(0{\cdot}2Py^+ - P + 1)]$;

Turbulent core $\Delta t^+ = 5[P - 1{\cdot}691 + \ln(5P + 1) + 0{\cdot}5 \ln y^+]$ (For $P < 0{\cdot}1$ use equation 13) the temperature profile the dimensionless presentation of Fig. 4, due to Squire [3] (extension of von Kármán [1]), can be used. It will be seen that no single equation in terms of y^+ represents the temperature profile, but that a family of curves is necessary for which the Prandtl number is the parameter. Introducing the dimensionless temperature, velocity and wall distance into equation (10), and substituting for u_m from equation (3), the ratio $\Delta t_m/\Delta t_r$ becomes

$$\frac{\Delta t_m}{\Delta t_r} = \frac{\Delta t_m^+}{\Delta t_r^+} = \frac{1}{\Delta t_r^+ I_1} \int_0^{r+} u^+ \Delta t^+ (r^+ - y^+)\, dy^+ = \frac{I_2}{\Delta t_r^+ I_1^+} \tag{11}$$

where

$$\left.\begin{aligned} I_2 = &\int_0^5 y^+ P y^+ (r^+ - y^+)\, dy^+ + \\ &+ \int_5^{30} (-3\cdot05 + 5 \ln y)\, 5\, [P + \\ &+ \ln (0\cdot2 P y^+ - P + 1)]\, (r^+ - y^+) dy^+ + \\ &+ \int_{30}^{r+} (5\cdot5 + 2\cdot5 \ln y^+)\, 5\, [P - 1\cdot691 + \\ &+ \ln (5P + 1) + 0\cdot5 \ln y^+] \times \\ &\times (r^+ - y^+) dy^+ \end{aligned}\right\} \tag{12}$$

I_2 must be solved by numerical integration for a series of values of r^+ and P. Since a value of R_m can be found for every value of r^+ as explained in Section 1.1, $\Delta t_m/\Delta t_r$ can finally be determined as a function of R_m and P. This procedure is extremely laborious, but Martinelli [4] obtained values of $\Delta t_m/\Delta t_r$ by a method equivalent to the one presented here, and his results are reproduced in Fig. 5. For comparison, $\Delta t_m/\Delta t_r$ for fully developed laminar flow is approximately 0·6.

It will be seen that the curves in Fig. 5 extend to the low Prandtl numbers which are associated with liquid metals. Equation (12) does not apply when P is much less than 0·1, and the case of liquid metals deserves special mention. Low P (i.e. high thermal conductivity k) implies a steeper temperature gradient in the turbulent core, and the equation for $y^+ > 30$ given in Fig. 4 is no longer applicable. (The equation is derived on the assumption that the thermal diffusivity ν/P is negligible compared with the eddy diffusivity ϵ and this is no longer true for the turbulent core when P is small.) The appropriate equation which must be used to replace the term

$$5[P - 1\cdot691 + \ln (5P + 1) + 0\cdot5 \ln y^+]$$

in the third integral of equation (12), for the limits $30 < y^+ < r^+$, is as follows

$$\left.\begin{aligned} \Delta t^+ = &\ 1\cdot25 \times \\ &\times \ln \left[\frac{(5/Pr^+) + (y^+/r^+)\{1 - (y^+/r^+)\}}{(5/Pr^+) + (30/r^+)\{1 - (30/r^+)\}}\right] + \\ &+ \frac{1.25}{\sqrt{\{1 + (20/Pr^+)\}}} \times \\ &\times \ln \left[\frac{\{(2y^+/r^+) - 1\} + \sqrt{\{1 + (20/Pr^+)\}}}{\{(2y^+/r^+) - 1\} - \sqrt{\{1 (+ 20/Pr^+)\}}} \times \right. \\ &\times \left. \frac{\{(60/r^+) - 1\} - \sqrt{\{1 + (20/Pr^+)\}}}{\{(60/r^+) - 1\} + \sqrt{\{1 + (20/Pr^+)\}}}\right] + \\ &+ 5\, [P + \ln (5P + 1)]. \end{aligned}\right\} \tag{13}$$

2.2. y_{mt}/r method

The mean temperature can be found by placing the probe at the appropriate wall distance y_{mt}, which in general is not the same as y_{mu}. y_{mt}/r $(= y_{mt}^+/r^+)$ can be found by substituting the appropriate value of Δt_m^+ in the equation for the turbulent core

$$\Delta t_m^+ = 5\, [P - 1\cdot691 + \ln (5P + 1) + 0\cdot5 \ln y_{mt}^+].$$

(For fluids of $P < 0\cdot1$ equation (13) would have to be used.) Δt_m^+ at any value of R_m can be obtained from both Δt_r^+ and the ratio $\Delta t_m/\Delta t_r$ taken from Fig. 5. Unfortunately Martinelli's data presented in Fig. 5 are not sufficiently detailed to enable reliable and consistent curves to be obtained for y_{mt}/r. Nevertheless these calculations have been made for fluids having Prandtl numbers greater than 0·7 and certain conclusions can be drawn from them.

(a) y_{mt}/r is not very sensitive to Reynolds number R_m.

(b) y_{mt}/r varies significantly with P, being approximately 0·31 at $P = 1$, 0·22 at $P = 10$, and 0·15 at $P = 40$.

(c) An analysis to reveal the permissible tolerance of positioning, analogous to that presented for velocity u_m in Section 1.3, has been carried out. For an error of $\delta \Delta t/\Delta t_m = 1$ per cent this gave

$$\frac{\delta y}{r} = \frac{\delta y^+}{r^+} = 0{\cdot}004\,\frac{y^+_{mt}}{r^+}\,\Delta t^+_m \qquad (14)$$

At $P = 1$, positioning is sensitive to $\delta y/r$ of 2–3 per cent (depending somewhat on R_m), and it becomes less sensitive at higher values of P; at $P > 50$ the probe will give a reading to better than 2 per cent anywhere between the axis and the buffer layer.

2.3. *Conclusions*

Mean temperature deduced from axial measurement is to be preferred to that deduced from a measurement at y_{mt}, for the same

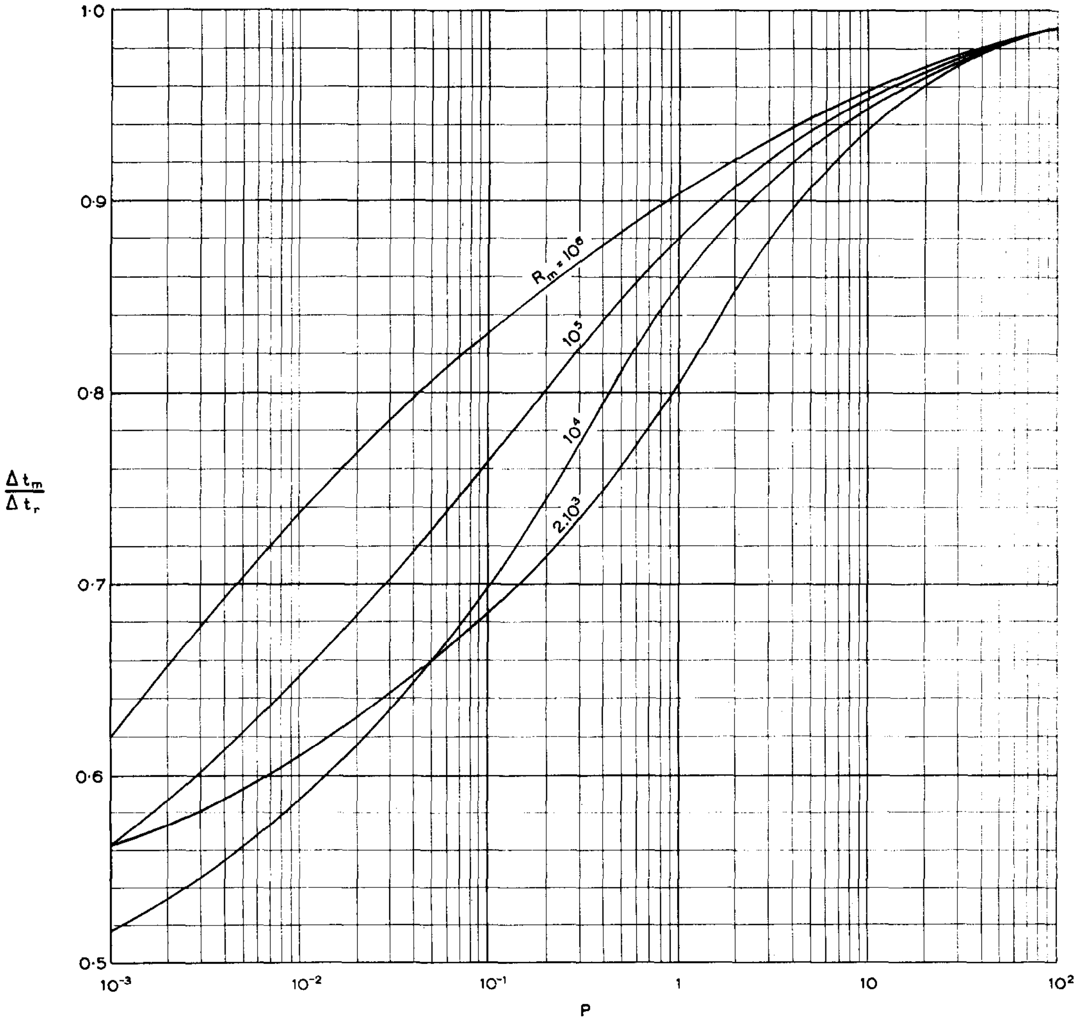

FIG. 5. Ratio $\Delta t_m/\Delta t_r$ against Prandtl number P, for several values of Reynolds number R_m (from Martinelli [4]).

reasons given in Section 1.3 when discussing the measurement of mean velocity. Only an approximate guide to positioning for the y_{mt}/r method can be given at present.

It must be remembered that the universal temperature profile has been derived by von Kármán using two principal assumptions.

(a) The ratio q/τ is independent of y (i.e. $q/\tau = q_w/\tau_w$.

(b) The ratio of eddy diffusivities for momentum and heat transfer, ϵ_M/ϵ_H, is equal to unity.

Assumptions (a) and (b) provide the basis for the simple Reynolds Analogy between momentum and heat transfer. The importance of assumption (b) has been tested by a few spot calculations to see its effect on the results of this section. By taking the lowest experimental value of ϵ_M/ϵ_H for flow in tubes ($0{\cdot}7 < \epsilon_M/\epsilon_H < 1$), and using a temperature profile modified to take account of variations in ϵ_M/ϵ_H, it was found that ϵ_M/ϵ_H has little effect on the ratio $\Delta t_m/\Delta t_r$.

It must be remembered that there are more serious deficiencies in the theory presented here. First, the universal velocity profile itself is an approximation and it fits well only in isothermal flow. ("Isothermal flow" implies a small value of $\Delta t = t - t_w$ so that it is reasonable to assume constant fluid properties over the flow cross-section.) In non-isothermal flow ν may vary appreciably across the tube. To allow for such variation in generalized and condensed form for all kinds of fluids is impossible. Deissler [5, 6] has done some work of this kind, allowing for variations of fluid properties with temperature, for air, supercritical water, and fluids of $P = 1$.

Secondly, the universal profiles are only good approximations for flow in hydraulically smooth tubes. The best check as to whether a tube is smooth is by comparing measured friction factors with those obtained from published data. Alternatively, a fairly safe criterion is that the surface roughness should be less than that corresponding to $y^+ = 5$, i.e. the roughness should lie within the laminar sublayer.

3. THERMOCOUPLE CONDUCTION ERROR

In order to reduce the error due to conduction along the wire when measuring temperature with a thermocouple, it is usual to lay the wire along an isothermal before taking it out radially, as shown in Fig. 6 (a). Often, particularly with tubes of small diameter, it is more convenient to use a centrally placed butt junction with taut wires emerging radially, an arrangement which gives rigidity and accurate location without elaborate supports that may interfere with the flow. It is necessary, however, to be able to estimate whether the observed couple temperature t_c reads the axial temperature t_r with negligible error, i.e. whether $(t_r - t_c)$ is small compared with $(t_r - t_w)$ (see Fig. 6 b).

So that a fair idea of the errors may be obtained, half the wire—from wall to junction—is treated as a "fin" projecting from a surface at t_w into a fluid at t_r, as shown in Fig. 6 (c).

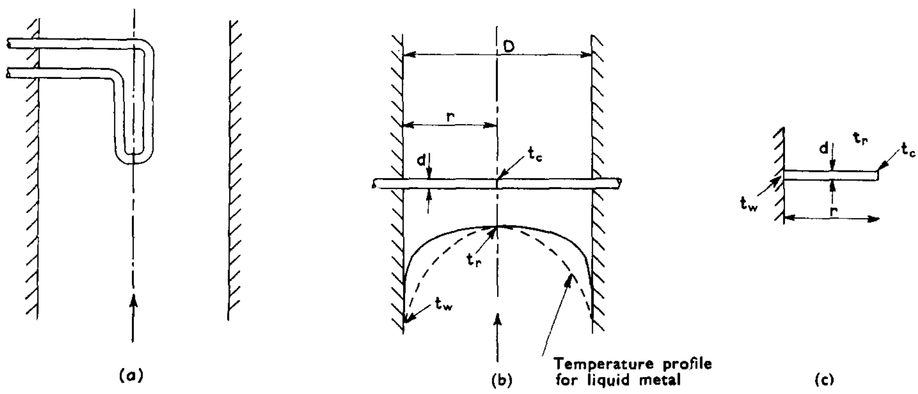

FIG. 6.

Simple fin theory shows, e.g. in Ref. [7], that

$$\frac{t_r - t_c}{t_r - t_w} = \frac{1}{\cosh mr} \tag{15}$$

where

$$mr = \left(\frac{h \times \text{fin circumference}}{k_d \times \text{fin cross-sectional area}}\right)^{1/2} r = \left(\frac{4h}{k_d d}\right)^{1/2} r \tag{16}$$

The principal assumptions in the derivation of equation (15) are:

(a) The fluid temperature t_r is uniform along the "fin".
(b) The heat transfer coefficient h is constant over the "fin" surface.
(c) The heat flow from the tip of the "fin" is negligible, i.e. $dt/dy = 0$ at $y = r$.

Assumption (a) is obviously not satisfied in a tube, but since for most fluids in turbulent flow the temperature profile is very flat across the turbulent core, this assumption is far off the mark only in a very thin layer near the wall. The only exceptions are liquid metals, for which t varies significantly in the core, and they are dealt with later.

Assumption (b) is also not satisfied in the tube because h depends mainly on the velocity u. But again the velocity profile is very flat in the turbulent core and this assumption is valid except in a very thin layer near the wall.

Assumption (c) is true only if the two metals forming the junction have the same thermal conductivity, because otherwise conditions are asymmetrical. With metals of different conductivity there will be a slight flow of heat from the better to the worse conductor but, because of the flatness of the temperature profile near the centre of the tube, this will be very small.

In the following, all three assumptions will be taken as adequate for determining the circumstances in which the error $(t_r - t_c)$ is negligible, even though they may not be quite satisfactory in estimating accurately errors when they are significant.

To evaluate equation (16) it is necessary to know how h varies with Reynolds number and Prandtl number for a cylinder in cross-flow. For air and other gases with similar Prandtl number, the curve on p. 259 of McAdams [8] has been used. This relates the Nusselt number $N_d = hd/k_f$ to the Reynolds number $R_d = u_m d/\nu_f$. For liquids (other than liquid metals) the following relation from p. 268 of McAdams [8] has been used:

$$\frac{N_d}{P^{0 \cdot 3}} = 0{\cdot}35 + 0{\cdot}56\, R_d^{0 \cdot 52} \tag{17}$$

Substituting for $h = N_d k_f/d$ in equation (16), the following is obtained

$$mr = \left(\frac{4N_d k_f}{d^2 k_d}\right)^{1/2} r = N_d^{1/2} \left(\frac{k_f}{k_d}\right)^{1/2} \left(\frac{D}{d}\right) = \mathrm{f}\,(R_d, P_f) \left(\frac{k_f}{k}\right)^{1/2} \left(\frac{D}{d}\right) \tag{18}$$

For any fluid of given Prandtl number P, it is therefore possible to express $(t_r - t_c)/(t_r - t_w)$ as a function of R_d, (D/d) and (k_d/k_f). Furthermore, since the Reynolds number R_m based on the tube diameter D is equal to $R_d(D/d)$, it is possible to substitute R_m for R_d.

The results of a few calculations are shown in Figs. 7(a)–(c). Figs. 7(a) and (b) are based on data for air, but they are equally applicable to other gases because all gases have Prandtl numbers which do not depart appreciably from the value for air ($P \simeq 0{\cdot}7$). Fig. 7(a) refers to a k_d/k_f ratio of 15,000 which corresponds approximately to a copper wire in air (e.g. a copper–constantan thermocouple). Fig. 7(b) refers to a k_d/k_f ratio of 1000 which corresponds to a manganin–constantan or chromel–alumel thermocouple in air, all of which metals have similar conductivities. It is clear that if errors of less than 1 per cent are desired, it is necessary to use a low conductivity couple, such as manganin–constantan, unless a very high D/d ratio can be used.

Fig. 7(c) refers to a liquid of Prandtl number 5 (e.g. water at about 90°F) and a k_d/k_f ratio of 600 (e.g. copper in water). The error is negligible except for very low D/d ratios. Manganin–constantan in water would yield a k_d/k_f ratio of about 40 and the error is then negligible for all practical values of D/d. Liquids of higher Prandtl number (e.g. oils of $P \simeq 100$) are no problem, because N_d is appreciably greater and

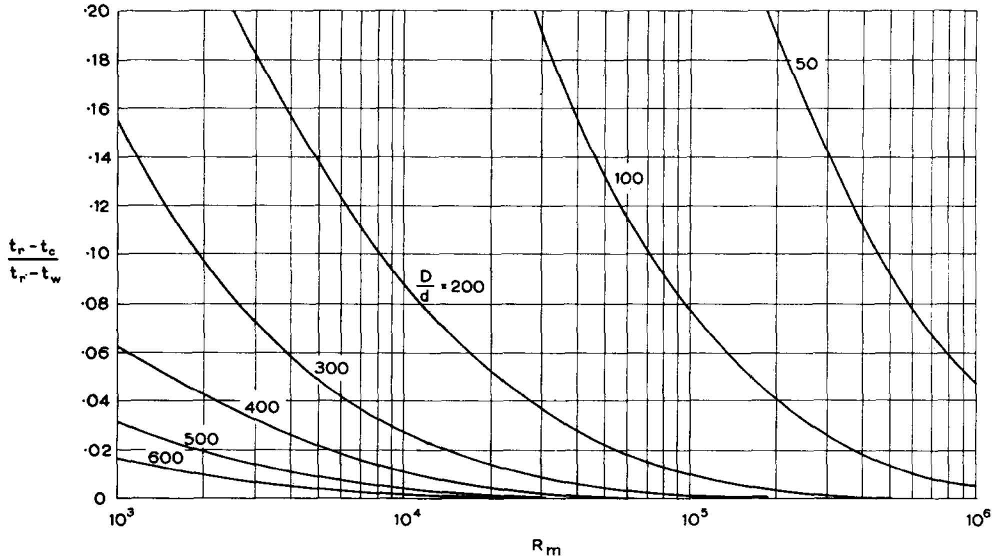

FIG. 7(a). Air and other gases ($P \simeq 0{\cdot}7$), $k_d/k_f = 15{,}000$.

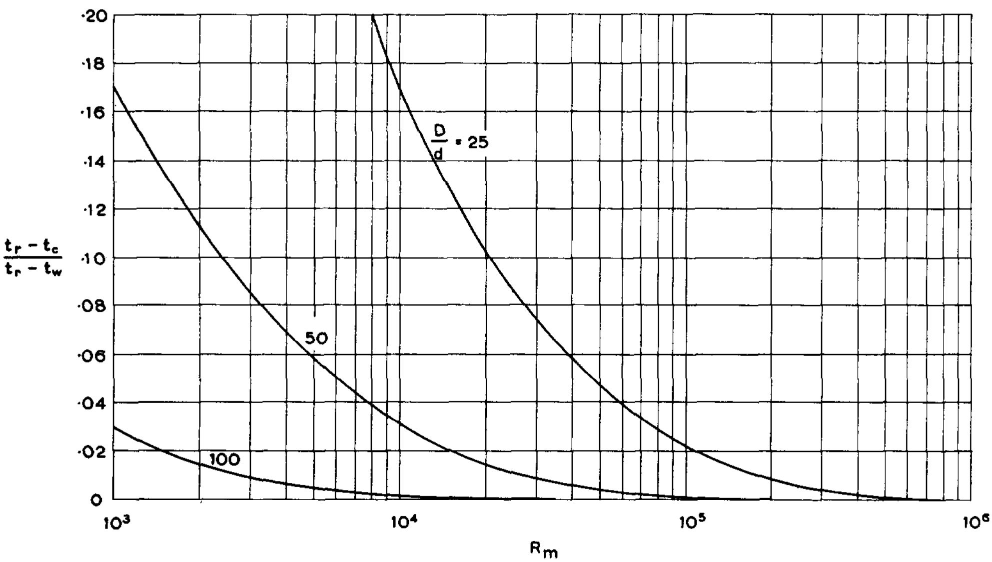

FIG. 7(b). Air and other gases ($P \simeq 0{\cdot}7$), $k_d/k_f = 1000$.

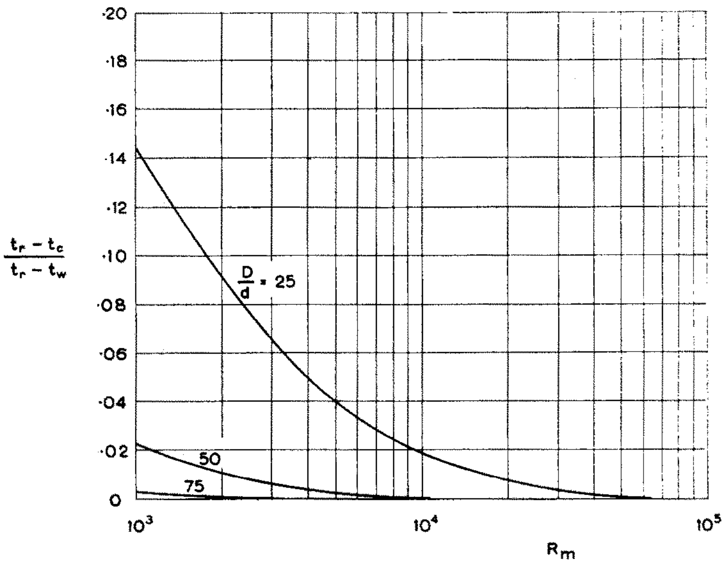

FIG. 7(c). Liquids of $P = 5$, $k_d/k_f = 600$.

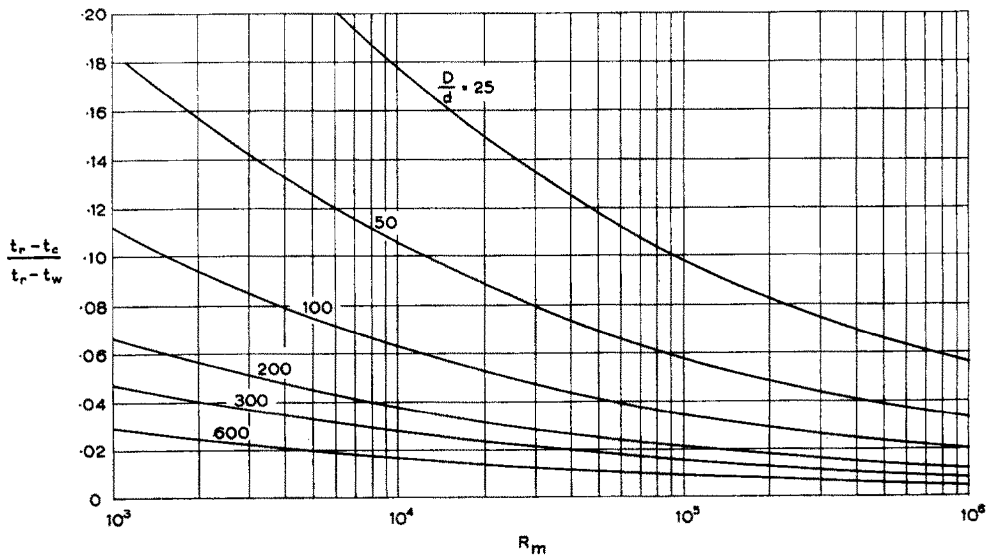

FIG. 8. Liquid metals of $P = 0{\cdot}01$, $k_d/k_f = 40$.

even the use of copper and low ratios of D/d results in negligible errors.

It is still necessary to consider liquid metals, which have very low Prandtl numbers (viz. 0·005–0·03). As already mentioned in Section 2.1, the high thermal conductivity results in a marked variation of temperature even in the turbulent core. Assumption (a) in the analysis of the "fin" is no longer valid, and a much better approximation is to assume that the fluid temperature varies linearly from t_w at the wall to t_r at the axis of the tube. With this modification it is possible to show that

$$\frac{t_r - t_c}{t_r - t_w} = \frac{1}{mr} \tanh mr \qquad (19)$$

mr is again given by equation (18), but N_d must be found from the appropriate equation for liquid metals

$$N_d = 1{\cdot}015\,(R_d\,P)^{1/2} \qquad (20)$$

taken from Eckert and Drake [9]. The results for a liquid metal of $P = 0{\cdot}01$ and $k_d/k_f = 40$ are given in Fig. 8. $k_d/k_f = 40$ corresponds approximately to copper wire in mercury. Even with liquid sodium, which has a much higher thermal conductivity than mercury, the use of copper would lead to too large an error. Manganin–constantan thermocouples would, however, be quite satisfactory. It is found that for all practical values of mr, $\tanh mr$ is virtually unity. Hence the results of Fig. 8 can easily be applied to any k_d/k_f ratio, because for any given D/d

$$\frac{t_r - t_c}{t_r - t_w} \propto \frac{1}{mr} \propto \left(\frac{k_d}{k_f}\right)^{1/2}$$

In the foregoing analysis it has been assumed that the wires are either bare, or carry a very light electrical insulation (e.g. a thin layer of enamel) of negligible thermal resistance. With liquid metals a rather heavier insulation might be necessary, in which case the estimates of error are on the low side.

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